The vanishing topology of ICMC2 singularities

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joint work with Matthias Zach, work in progress

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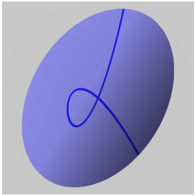
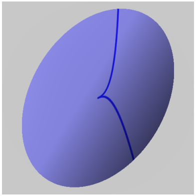
Leibniz Universität Hannover

Kaiserslautern, 12. Oktober 2016

Some Hypersurface Singularities



$V(y^2 - x^2 - x^3)$



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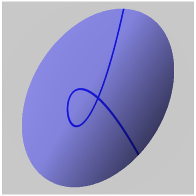

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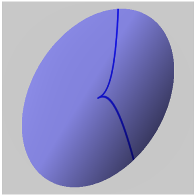

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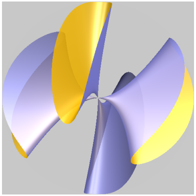
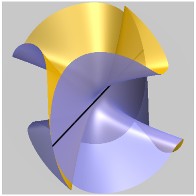
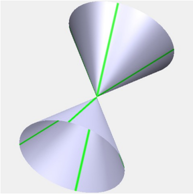


$V(x^2 + y^4 - z^4 - 3x^2y^2)$

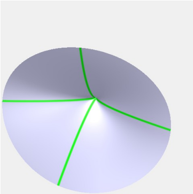


$V(x^2 - y^4 + x^2z^4)$

Higher Codimension: Space Curves



$V(xy, x^2 + y^2 - z^2)$



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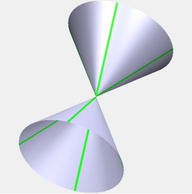

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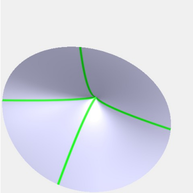

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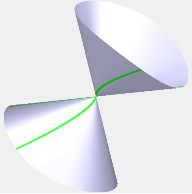
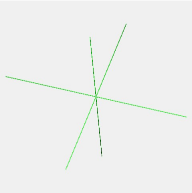


$V(x^2 - yz, z^3 - xy)$



$V(xy, xz, yz)$

Hypersurfaces, Complete Intersections . . .

Hypersurface singularities:

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But on the previous slide: 3 lines in $\mathbb{C}^3$

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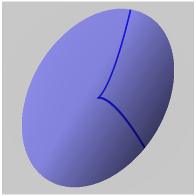
non-trivial relations:

$$\begin{aligned} z\cdot xy \qquad\qquad - x\cdot yz &= 0 \\ z\cdot xy - y\cdot xz \qquad\qquad &= 0 \end{aligned}$$

Which singularity is 'less singular'?



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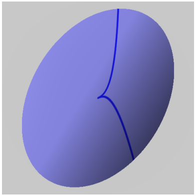

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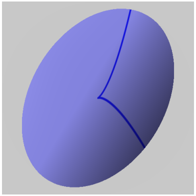

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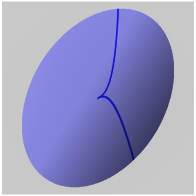

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First idea of comparison: Consider ideals

$$\langle f \rangle, \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle, \langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \subset \mathbb{C}\{x, y\}$$

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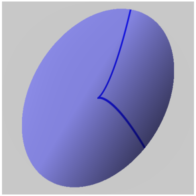

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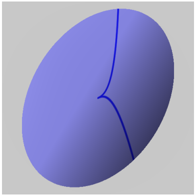

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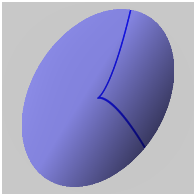

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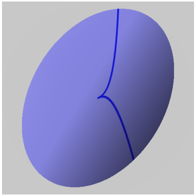

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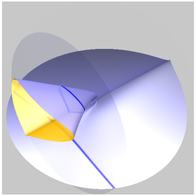
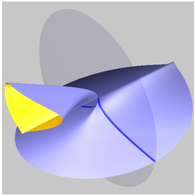
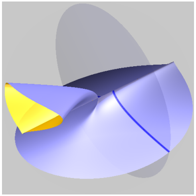
Advantage: Use of tools of commutative algebra!

Deformations in pictures

$y^2 - x^5 - t \cdot x^3$:

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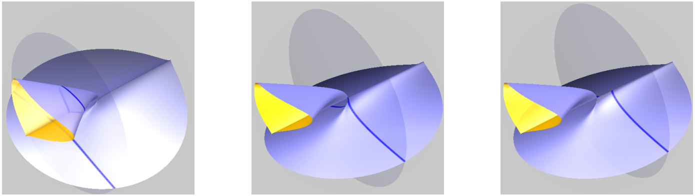

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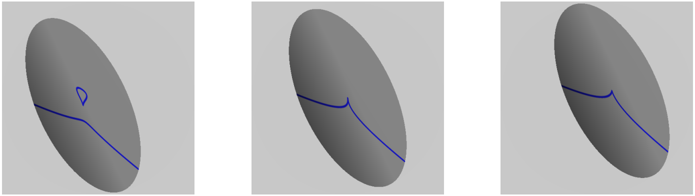
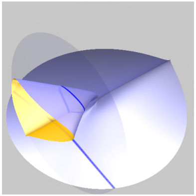
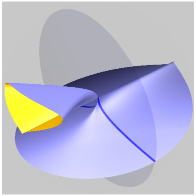
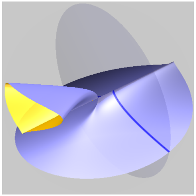


Deformations – more formally

More generally:
A deformation of $(X,0)$ over a base $(S,0)$

$$\begin{array}{ccc} (X,0) & \hookrightarrow & (\mathcal{X},0) \\ \downarrow & & \downarrow \text{flat} \\ \{0\} & \hookrightarrow & (S,0) \end{array}$$

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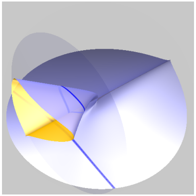

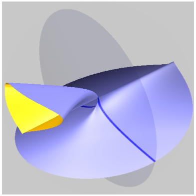

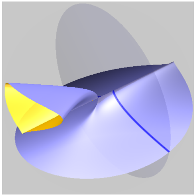

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Flatness: relation lifting property

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Given $(X, 0)$ as special fibre:

- What singularities can appear in the family?
- Is there a family containing all of these?

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simple := only finitely many types in versal family

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Milnor-algebra of a hypersurface singularity $(V(f), 0)$:

$$M(f) := \mathbb{C}\{\underline{x}\}/\langle \frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n} \rangle$$

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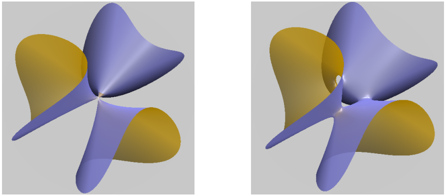
geometric interpretation using a smoothing:



For complete intersections: Lê-Greuel Formel

topological type of a bouquet of μ spheres of $dim(X)$!

Simple singularities

- hypersurface singularities: A-D-E singularities [Arnold, 1972]
- ICIS: fat points in the plane, space curves [Giusti, 1983]

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beginning of Arnold's classification:

curve	A_1	A_2	A_3 $y^2-x^4=0$	A_4	D_4 $x^3-y^3=0$
μ	1	2	3	4	4
τ	1	2	3	4	4

Simple Cohen-Macaulay codimension 2 singularities (simple ICMC2)

Simple isolated, non-ICIS singularities [FK1999], [FK,Neumer2010]:

- fat points in the plane
- space curves
- normal surface germs in $(\mathbb{C}^4, 0)$
  (coincides w.Tjurina's list of rational triple points)
- germs of 3-folds in $(\mathbb{C}^5, 0)$
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With exception of 4-folds: all smoothable.

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- free resolution of the form

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special property of this case – Pinkham's example!

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- determinantal singularities: 'singular Milnor number' [Damon-Pike 2014]
- normal surface singularities [Wahl 2013]

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For hypersurfaces and ICIS: $\mu \geq \tau$

beyond ICIS:

- in general: no longer a bouquet of spheres
- e.g. Betti numbers b_2 and b_3 may be non-zero for 3-folds

Known results:

- isolated curve singularities
- isolated determinantal surface singularities [Ruas,daSilva Pereira 2014; Nuno-Ballesteros,Orefice-Okamoto,Tomazella 2013]
- determinantal singularities: 'singular Milnor number' [Damon-Pike 2014]
- normal surface singularities [Wahl 2013]

'Milnor number' for determinantal singularities

For simple ICMC2 surfaces and bounding cases:

$$\tau - \mu = 1$$

[da Silva Pereira; Damon-Pike] by explicit computation,
[Wahl 2013] for surface singularities

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Damon and Pike's Observation for 3-folds:

- negative 'Milnor numbers' occur
  but in computed examples: only -1 Why?
- 'Milnor number' stays constant in some families with increasing Tjurina number

A 3-fold example

family of simple isolated 3-fold singularities

$$\begin{pmatrix} w & y & x \\ z & w & y + v^k \end{pmatrix}$$

Tjurina number: $\tau = 2k - 1$

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Tjurina number: $\tau = 2k-1$

'Milnor number': $\mu = -1$ (constant and negative!)

We know: $\mu = b_3 - b_2$
($b_1 = 0$ according to [Greuel-Steenbrink 1983])

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Known construction for ICMC2 surface singularities
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The setting:
Let $V_1 \subset \mathbb{C}^{t(t+1)}$ the generic maximal minor determinantal singularity (t columns, $t+1$ rows).
Row vectors span $t-1$-dimensional hyperplane $P_A \subset \mathbb{C}^t$

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$\Longrightarrow$ rational map:

$$\begin{array}{rcl} P : V_r & \dashrightarrow & \mathrm{Grass}(t-1,t) \\ A & \mapsto & P_A \end{array}$$

Tjurina modification II

resolving indeterminacy provides

$$\begin{array}{ccc} W_1 & & \\ \downarrow \pi & \searrow^{\hat{P}} & \\ V_1 & \overset{P}{\dashrightarrow} & \mathrm{Grass}(t-1, t) \end{array}$$

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Now considering M as a map from a given ICMC2 X_0 into V_1:

$$\begin{array}{ccccc} Y_0 := X_0 \times_{V_1} W_1 & \xrightarrow{\hat{M}} & W_1 & & \\ \downarrow \pi & & \downarrow \rho & \searrow^{\hat{P}} & \\ X_0 & \xrightarrow{M} & V_1 & \overset{P}{\dashrightarrow} & \mathbb{P}^r \end{array} \tag{1}$$

Tjurina modification III

Equations for Y_0:

$$M \cdot \begin{pmatrix} s_1 \\ \vdots \\ s_t \end{pmatrix} = 0$$

locally complete intersection

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Construction also works in family:

$$\begin{array}{ccccccc}
X_0 \times_{V_1} W_1 & \hookrightarrow & X \times_{V_1} W_1 & \xrightarrow{\hat{M}} & W_1 & & \\
\downarrow \pi_0 & & \downarrow \pi & & \downarrow \rho & \searrow \hat{P} & \\
X_0 & \hookrightarrow & X & \xrightarrow{M} & V_1 & \overset{P}{\dashrightarrow} & \mathbb{P}^r \\
\downarrow & & \downarrow \varepsilon & & & & \\
\{0\} & \longrightarrow & \mathbb{C} & & & &
\end{array} \qquad (2)$$

π is an isomorphism for non-singular fibres.

The structure of Y_0 for $t = 2$

Exceptional locus is a $\mathbb{P}^1$.

Two situations can occur:

(A) Y_0 has only ICIS (at fin. many points of $\mathbb{P}^1$) iff

$$M \sim \begin{pmatrix} * & x_1 \\ * & x_2 \\ * & x_3 \end{pmatrix}$$

(B) Y_0 is singular along the whole $\mathbb{P}^1$ otherwise

The topology of the Milnor fibre, case (A)

Theorem (FK,Zach 2015)

Given a smoothing $X_0 \hookrightarrow X \xrightarrow{\varepsilon} \mathbb{C}$ of an ICMC2 singularity in $(\mathbb{C}^5, 0)$ and a Tjurina modification as above in case (A), the Betti numbers of the Milnor fibre are given by

$$b_0 = 1, \quad b_1 = 0, \quad b_2 = 1, \quad b_3 = r$$

where $r \in \mathbb{N}$ is the sum of the Milnor numbers of the ICIS of Y_0.

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analogously $b_2 = b_2(Y) + 1$ for surfaces in $(\mathbb{C}^4, 0)$.

The topology of the Milnor fibre, case (B)

Theorem (Zach, 2016)

Given a smoothing $X_0 \hookrightarrow X \xrightarrow{\varepsilon} \mathbb{C}$ of an ICMC2 singularity in $(\mathbb{C}^5, 0)$ and a Tjurina modification as above in case (B), the second Betti number of the Milnor fibre is 1.

analogously: $b_2 = \mu_Y + 1$ for surfaces in $(\mathbb{C}^4, 0)$.

- non-isolated singularity along a $\mathbb{P}^1$
- transversal type of this singularity relevant
- contribution by 'vertical monodromy'
- further 'worse' singularities may sit at points of the $\mathbb{P}^1$